INFLUENCE OF THE THERMOCAPILLARY EFFECT ON THE SHAPE OF A FREE BOUNDARY AT HIGH MARANGONI NUMBERS

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1. We propose to investigate the steady-state problem of viscous, thermally conducting fluid flow in an unbounded domain D under the influence of thermocapillary forces produced by nonuniform heating of a free boundary Γ for small values of the viscosity coefficient $\nu \to 0$ and thermal diffusivity $\chi \to 0$:

$$(\mathbf{v}, \nabla)\mathbf{v} = -(1/\rho)\nabla p + \nu\Delta\mathbf{v} + \mathbf{g},\ \mathbf{v}\nabla T = \chi\Delta T,\ \operatorname{div}\mathbf{v} = 0; \tag{1.1}$$

$$\begin{gathered} p = 2\nu\rho\mathbf{n}\Pi\mathbf{n} + \sigma(k_1 + k_2) + p_*, \quad (x, y, z) \in \Gamma, \\ 2\nu\rho\Pi\mathbf{n} - 2\nu\rho(\mathbf{n}\Pi\mathbf{n})\mathbf{n} = \nabla_1\sigma; \quad \mathbf{v}\mathbf{n} = 0, \quad (x, y, z) \in \Gamma, \\ \varkappa\partial T/\partial n = q,\ (x, y, z) \in \Gamma_1;\ T = T_\Gamma,\ (x, y, z) \in \Gamma_2, \\ \mathbf{v} = 0,\ \partial T/\partial n = 0,\ (x, y, z) \in L. \end{gathered} \tag{1.2}$$

Here $\mathbf{v} = (v_x, v_y, v_z)$ is the velocity vector, T is the temperature, $\mathbf{g} = -g\mathbf{e}_z$ [$\mathbf{e}_z = (0, 0, 1)$ is the unit vector along the z axis, and g is the gravitational acceleration], $\mathbf{n}$ is the unit vector outward normal to the free boundary Γ, Π is the strain-rate tensor, k_1 and k_2 are the principal curvatures of the surface Γ, $p_* =$ const is the pressure on Γ, $\nabla_1 = \nabla - (\mathbf{n}\nabla)\mathbf{n}$ is the gradient along Γ, σ is the coefficient of surface tension, which is assumed to be a linear function of the temperature: $\sigma = \sigma_0 + \sigma_T(T - T_*)$ (σ_0, σ_T, and T_* are known constants, $\sigma_T < 0$), and L is a solid boundary. The surface I consists of two parts Γ_1 and Γ_2, q(x, y, z) is the specified heat flux onto Γ_1, and κ is the thermal conductivity. The velocity field vanishes at infinity.

Nonlinear boundary layers are formed near the boundaries of the domain for vanishingly small viscosity and thermal conductivity. The flow is approximately described by the Euler equations everywhere outside the boundary layers in the unbounded domain. Several authors [1-5] have investigated nonlinear Marangoni boundary layers formed near a free boundary as a result of the thermocapillary effect. Asymptotic expansions have been derived [6] in the limit $\nu \to 0$ for the solution of the steady-state problem of fluid flow under the influence of tangential stresses.

Here we investigate the formal asymptotic expansions of the solution of problem (1.1), (1.2) in the limit $\nu, \chi \to 0$. We reduce the problem to dimensionless form and introduce the small parameter $\varepsilon = M^{-1/3}$ ($M = |\sigma_T| dQ\rho^{-1}\nu^{-2}\kappa^{-1}$ is the Marangoni number, and d and Q are characteristic scales of length and heat flow). For the dimensionless pressure p' we have the relation $p = Pp' - \rho gz$ ($P = \sigma_0/d$ is a pressure scale). The characteristic velocity in the boundary layer near the free boundary $U = (\sigma_T^2 Q^2 \nu^{-1} d^{-1} \kappa^{-2} \rho^{-2})^{1/3}$ is adopted as the velocity scale. Asymptotic expansions of the solution of problem (1.1), (1.2) are constructed in the form

$$\begin{gathered} \mathbf{v} \sim \mathbf{h}_0 + \varepsilon(\mathbf{h}_1 + \mathbf{v}_1 + \mathbf{w}_1) + \ldots,\ p' \sim \lambda q_0 + \varepsilon\lambda(p_1 + q_1 + r_1) + \ldots, \\ T \sim \theta_0 + T_0 + t_0 + O(\varepsilon),\ \zeta \sim \zeta_0 + \varepsilon\zeta_1 + \ldots, \end{gathered} \tag{1.3}$$

where $\lambda = |\sigma_T| Q\sigma_0^{-1}\kappa^{-2}$ is the capillary constant [3], and $z = \zeta(x, y)$ is the equation for the free boundary. We denote by D_Γ the domain of the boundary layer near the free boundary, and by D_L the domain near the solid wall. Then $\mathbf{h}_k$, q_k, and θ_0 are functions representing the solutions of the boundary-layer problem in D_Γ; $\mathbf{w}_1$, r_1, and t_0 are the same in D_L, and $\mathbf{v}_1$, p_1, and T_0 characterize the solution outside D_L and D_Γ. The orders of the principal terms in the expansions (1.3) are determined from the conditions that the viscous and inertial terms in the Navier–Stokes system (1.1) and in the boundary conditions (1.2) for the tangential stresses are of the same order. The thickness of the boundary layer in D_Γ is of the order of ε in this case.

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2. The boundary-value problem for the principal terms of the asymptotic expansion (1.3), which govern the flow in the boundary layer near the free boundary, is obtained by applying the second iteration process of the Vishik–Lyusternik method [7] to the system (1.1), (1.2). Near the surface Γ we introduce local orthogonal coordinates ξ, φ, θ [6] [ξ is the distance from the point N(x, y, z) to Γ, and φ, θ are the curvilinear coordinates on Γ of the base of the normal dropped from the point N onto Γ]. We assume that segments of the normals to Γ do not intersect for sufficiently small ξ.

Let $h_{\varphi k}$, $h_{\theta k}$, and $h_{\xi k}$ be the components of the vector $\mathbf{h}_k$ in the local coordinates. We adopt the function $T_0 = 0$ as the solution of the degenerate problem for (1.1), (1.2) when $\nu = \chi = 0$. We substitute Eq. (1.3) in (1.1) and (1.2), form the Taylor series expansion of $\mathbf{v}_1$ and p_1 in powers of ξ, and introduce the extension transformation $\xi = \varepsilon s$. We denote $H_{\xi 1} = h_{\xi 1} + \mathbf{v}_1\mathbf{n}|_\Gamma$. Setting the coefficients of ε^{-1} and ε_0 equal to zero, we obtain $h_{\xi 0} = 0$ and find that $h_{\varphi 0}$, $h_{\theta 0}$, and $H_{\xi 1}$ satisfy the Prandtl boundary-layer equations. We state the boundary-value problem for $h_{\varphi 0}$, $H_{\xi 1}$, and θ_0 in the planar case, regarding the coordinate φ as the length of the train along the surface Γ:

$$\begin{gathered} h_{\varphi 0}\partial h_{\varphi 0}/\partial\varphi + H_{\xi 1}\partial h_{\varphi 0}/\partial s = \partial^2 h_{\varphi 0}/\partial s^2, \\ h_{\varphi 0}\partial\theta_0/\partial\varphi + H_{\xi 1}\partial\theta_0/\partial s = \mathrm{Pr}^{-1}\partial^2\theta_0/\partial s^2,\ \partial h_{\varphi 0}/\partial\varphi + \partial H_{\xi 1}/\partial s = 0, \\ \partial h_{\varphi 0}/\partial s = -\partial\theta_0/\partial\varphi, \quad \partial\theta_0/\partial s|_{\Gamma_1} = -\mathrm{Pr}^{-1/2}q, \quad \theta_0|_{\Gamma_2} = \theta_a, \quad H_{\xi 1} = 0 \quad (s = 0), \\ h_{\varphi 0} = h_{\xi 1} = \theta_0 = 0 \quad (s = \infty) \end{gathered} \tag{2.1}$$

[Pr is the Prandtl number, and $q(\varphi)$ is the given dimensionless heat flux onto Γ_1]. We note that problem (2.1) for $\theta_0 = 0$ and for specified stresses on Γ and a given initial velocity profile in the domain D has been investigated previously by Kuznetsov [5], who gives the conditions for its solvability.

We determine the value of q_0 on the free boundary. Following [6], we derive an equation for q_0 in D_Γ:

$$q_0 = -k_1\int_s^\infty h_{\varphi 0}^2\,ds - k_2\int_s^\infty h_{\theta 0}^2\,ds. \tag{2.2}$$

Substituting Eq. (2.2) in the boundary condition (1.2) for the normal stresses and letting ν tend to zero, we obtain the free-boundary equation in the principal approximation (in dimensioned form)

$$\sigma(k_1 + k_2) + k_1\int_0^\infty h_{\varphi 0}^2\,ds + k_2\int_0^\infty h_{\theta 0}^2\,ds = \rho g z + c. \tag{2.3}$$

This equation is readily simplified for the planar problem. In this case, integrating the first equation (2.1) initially with respect to s on the semiaxis $(0, \infty)$ and then with respect to φ on the interval $[\varphi_0, \varphi]$, we have the relation

$$\int_0^\infty h_{\varphi 0}^2\,ds = [\theta_0(\varphi) - \theta_0(\varphi_0)]_{s=0} + \int_0^\infty f_0^2(s)\,ds \tag{2.4}$$

($f_0(s) = h_{\varphi 0}(s, \varphi_0)$ is the velocity profile in the cross section $\varphi = \varphi_0$). We substitute Eq. (2.4) in (2.3), assume that $k_2 = 0$, and, writing the dimensionless coefficient of surface tension in the form $\sigma = 1 - \lambda\theta_0|_\Gamma$, derive the free-boundary equation in the dimensionless form

$$k_1\left[1 - 2\lambda T_\Gamma(\varphi) + \lambda T_\Gamma(\varphi_0) + \int_0^\infty f_0^2\,ds\right] = \mathrm{B}z + c, \tag{2.5}$$

where T_Γ is the value of the temperature θ_0 on the free boundary, and $\mathrm{B} = \rho g d^2/\sigma_0$ is the Bond number. Thus, the function ζ_0 in Eq. (1.3) is determined by solving Eq. (2.3) or (2.5).

Inviscid flow outside D_L and D_Γ is obtained by applying the first iteration process [7] to Eqs. (1.1) and (1.2). The principal terms of the asymptotic expansions (1.3) of $\mathbf{v}_1$, and p_1 are determined by solving the boundary-value problem

$$(\mathbf{v}_1, \nabla)\mathbf{v}_1 = -\nabla p_1,\ \operatorname{div} \mathbf{v}_1 = 0,$$
$$\mathbf{v}_1\mathbf{n}_0|_{\Gamma_0} = H_{\xi 1}|_{\xi=\infty},\quad \mathbf{v}_1\mathbf{n}_1|_L = 0,\quad \mathbf{v}_1 = 0 \quad (x^2+y^2+z^2=\infty)$$

(Γ_0 is the surface $z = \zeta_0$, $\mathbf{n}_0$ is the vector normal to Γ_0, and $\mathbf{n}_1$ is the vector normal to the solid wall).

The vector function $\mathbf{w}_1$ describes the velocity field in the boundary layer near the solid wall and offsets the discrepancy induced by the fact that the vector $\mathbf{v}_1$ satisfies the no-slip condition on L. The boundary-value problem for $\mathbf{w}_1$ and r_1 is not given, because these functions contribute to the free-boundary equation only in higher approximations, beginning with the second.

3. We now consider the planar problem of calculating the shape of a free boundary of a capillary liquid which is poured onto a solid horizontal surface and sets it in the half-plane $x \geq 0$. We denote by β the contact angle formed by the liquid on the wetting line $x = 0$ with the solid wall. We assume that on the part of the free boundary Γ_1 a constant heat flux $q = \text{const}$ is specified at $\varphi \in [0, \ell]$ and a constant temperature is specified at $\varphi > l$ (φ is the arclength read from the contact point along the free boundary). We give the solution of the system of boundary-layer equations (2.1) on the part Γ_1 for $q = 1$. Self-similar solutions have been found [2] for the temperature boundary-layer equations near the interface of two immiscible liquids. The system (2.1) admits the solution $h_{\varphi 0} = f'(\eta)$, $\theta_0 = \sqrt{\varphi}\tau(\eta)$, $\eta = s/\sqrt{\varphi}$. For $f(\eta)$ and $\tau(\eta)$ we derive the boundary-value problem (it is not necessay to specify the initial profile, because it satisfies the self-similarity condition)

$$\begin{gathered}2f''' + ff'' = 0,\ 2\,\mathrm{Pr}^{-1}\tau'' + f\tau' - f'\tau = 0,\\ f(0) = 0,\ f''(0) = -0{,}5\tau(0),\ \tau'(0) = -\mathrm{Pr}^{-1/2},\ f'(\infty) = \tau(\infty) = 0.\end{gathered} \tag{3.1}$$

We integrate the resulting system numerically for various Prandtl numbers, using the Runge–Kutta method. For fixed values of Pr the functions $f'(\eta)$ and $\theta(\eta)$ decrease monotonically as η increases and for large η the decay of the velocity is greater for smaller numbers Pr. The velocity profiles for different numbers Pr intersect, but the temperature profiles do not. As Pr is increased, the thickness of the dynamic (velocity) boundary layer increases, and the thickness of the temperature boundary layer decreases. Figure 1 shows graphs of the velocity $f'(0)$ (curve 1) and the temperature $\tau(0)$ (curve 2) as a function of the Prandtl number on the free boundary.

We calculate the shape of the free boundary integrating Eq. (2.5), in which we set $\varphi_0 = 0$ and $f_0 = 0$ (since $h\varphi_0 = 0$ at $\varphi = 0$). Taking the solution of problem (3.1) into account, we write Eq. (2.5) in the form

$$\zeta_0''(1+\zeta_0'^2)^{-3/2}(1+\lambda F(\varphi)) = \mathrm{B}\zeta_0 + c \tag{3.2}$$

[the prime signifies differentiation with respect to $F(\varphi) = \tau(0)(2\sqrt{\varphi} - \sqrt{l})$ at $0 \leq \varphi \leq \ell$ and $F(\varphi) = \tau(0)\sqrt{l}$ for $\varphi > l$). The boundary conditions $\zeta_0'(0) = \tan\beta$, $\zeta_0(0) = 0$ (β is the contact angle) hold for problem (3.2). We integrate Eq. (3.2) numerically for various λ and β, $B = 1$, $\mathrm{Pr} = 7$, and $\tau(0) = 0.2235$. We set $\ell = 9$ in the calculations; the free boundary approaches a horizontal asymptote at this arclength. For the integration Eq. (3.2) is rewritten in parametric form with the arclength as its parameter. The unknown constant c is evaluated with the additional condition $\zeta_0'(\infty) = 0$. The thickness of the liquid layer at infinity is determined from the equation $H = -cB^{-1}$. The calculations are carried out for λ in the interval $0 \leq \lambda < \lambda_* = 1.4914$. At $\lambda = \lambda_*$ the coefficient of the leading derivative in Eq. (3.2) vanishes at $\varphi = 0$.

Figure 2 shows the contact angle β as a function of the parameter λ, which characterizes the heat flux. The layer thickness H is constant along curves 1-4 and has the values 1.85, 1.5, 1, and 0.5, respectively. The greater the thickness of the layer, the more rapidly β increases with λ. For each thickness H the maximum value of β is equal to π and is attained at some $\lambda_1 \leq \lambda_*$; for example, $\lambda_1 = 0.5$ for $H = 1.85$. The value of λ_1 increases as H decreases, and $\lambda_1 \to \lambda_*$ in the limit $H \to 0$. The calculated dependence of H on the angle β shows that H increases monotonically with β for a fixed λ and attains a maximum at $\beta = \pi$. This maximum is equal to two at $\lambda = 0$ and decreases to zero as increases.

These results can be used to calculate the shape of a flat meniscus adjacent to a solid vertical wall. The liquid now fills the infinite domain bounded by the wall $x = 0$ and the

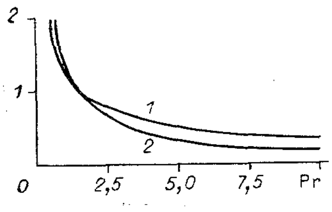

Fig. 1

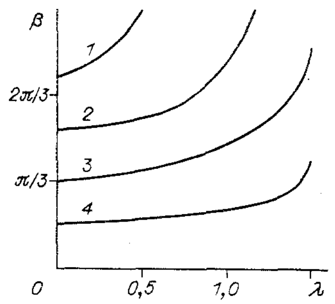

Fig. 2

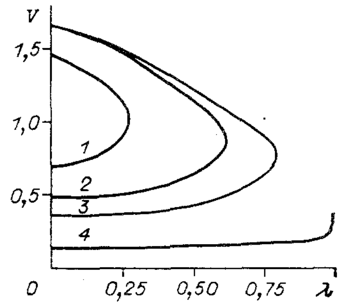

Fig. 3

free boundary $z = \zeta(x)$. Equation (3.1) is integrated under the conditions $\zeta_0'(0) = \tan\beta$, $\zeta_0(\infty) = 0$, where β is the acute angle between the tangent to the free boundary and the x axis at $x = 0$. The function $F(\varphi)$ is the same as before. The constant $c = 0$ in Eq. (3.2). For $0 \le \lambda < \lambda_*$ the height $h = \zeta_0(0)$ of the meniscus increases monotonically with β and attains a maximum at $\beta = \pi/2$. This maximum is equal to $\sqrt{2}$ at $\pi = 0$. For a fixed angle β the height of the meniscus decreases with increasing λ and becomes equal to zero at $\lambda = \lambda_* = 1.491$.

We note that Eq. (1.3) does not contain the boundary-layer functions, which show up in the domain D_1 as neighborhoods of a contact point of the free boundary and the solid wall. In D_1 the asymptotic expansions exhibit a more complex character than (1.12). The boundary-layer equations in D_1 coincide with the complete Navier–Stokes equations. The boundary-layer functions contribute to the free-boundary equation only in higher approximations and will therefore be disregarded. Asymptotic investigations of the Navier–Stokes system near a contact point are reported in [8, 9].

4. We now consider the axisymmetrical problem of calculating the shape of the free boundary of a droplet issuing from a hole in a plane horizontal wall. The free surface rests on the edge of a circular hole of radius R. We assume that the temperature distribution at the free boundary is given: $T - T_* = ARf(\varphi)$ (φ is the dimensionless arclength in the axial cross section, read from the symmetry axis). We introduce the parameter $\lambda = |\sigma_T|AR\sigma_0^{-1}$, which characterizes the temperature gradient along the free boundary. Because of axial symmetry, $h_{\theta 0} = 0$, we infer from the boundary-layer equations that $h_{\varphi 0}$ obeys the relation

$$\int_0^\infty h_{\varphi 0}^2\,ds = \frac{1}{r}\int_0^\varphi r(\varphi)\frac{d\sigma}{d\varphi}\,d\varphi \tag{4.1}$$

[$r = r(\varphi)$ and $z = z(\varphi)$ are the parametric equations for the free boundary in cylindrical coordinates]. Allowance is made here for the fact that $h_{\varphi 0} \to 0$ in the limit $\varphi \to 0$. The free boundary satisfies Eq. (2.3), which, according to relation (4.1), reduces in dimensionless form to the system of equations

$$\frac{(rz')'}{rr'}(1-\lambda f) - \lambda\frac{r'z''-r''z'}{r}\int_0^\varphi rf'\,d\varphi = z\mathrm{B} + c,\quad r'^2 + z'^2 = 1. \tag{4.2}$$

Placing the origin on the free boundary on the symmetry axis, we write the boundary conditions $r(0) = z(0) = z'(0) = 0$, $r'(0) = 1$. The system (4.2) is integrated numerically on the basis of the Runge–Kutta method. In the limit $\varphi \to 0$ the solution of the system is expanded as a power series in φ and is matched with the numerical solution. The Bond number, the parameter λ, and the function $f(\varphi)$ are given, and it is required to determine the constant c, the droplet volume V, and the angle β ($\pi - \beta$ is the angle at the contact point between the tangent to the axial cross section of the free boundary and the horizontal solid wall).

In the case $f(\varphi) = \exp(-\varphi)$ the calculations are carried out for B = −1, a constant volume V = 1, and various R. We note that several free boundary configurations with different angles β can exist for a fixed R. For the first branch of the solutions, the angle β decreases with increasing λ from a certain β_0 at $\lambda = 0$ to β_1 at $\lambda = \lambda_*$. A solution does not exist for $\lambda > \lambda_*$; λ_* depends on R and increases with the value of R, but does not exceed unity. For example, $\lambda_* = 0.6$, $\beta_0 = 87.6°$, and $\beta_1 = 60.8°$ for R = 0.975. For the second branch of the solutions β decreases with decreasing λ from β_1 at $\lambda = \lambda_*$ to a certain β_3 at $\lambda = 0$.

For $f(\varphi) = 1 - \varphi$ $(0 \leqslant \varphi \leqslant 1)$, $f(\varphi) = 0$ $(\varphi > 1)$ the calculations are carried out for constant R = 1, B = −1, and various volumes V. Several solutions can exist for a given V. For the first branch of the solutions at a fixed angle β the droplet volume increases with increasing λ from V_1 at $\lambda = 0$ to V_2 at $\lambda = \lambda_1$. A solution does not exist for $\lambda > \lambda_1$; λ_1 depends on β and increases from 0 to 1 as β decreases from 86° to 0. For the second branch of the solutions the droplet volume increases from V_2 at $\lambda = \lambda_1$ to V_3 at $\lambda = 0$. For example, $V_1 = 0.36$, $V_2 = 0.78$, $V_3 = 1.66$, and $\lambda_1 = 0.79$ at $\beta = 60°$.

The volume is plotted as a function of λ in Fig. 3. Curves 1-4 correspond to $\beta = 80°$, 70°, 60°, and 30°, respectively. The maximum value of λ is equal to unity and corresponds to the critical free-boundary temperature at which the coefficient of surface tension becomes equal to zero.

5. Here we discuss the influence of the thermocapillary effect on the shape of the free boundary of a liquid filling a horizontal layer, whose thickness is of the order of the boundary layer thickness ε. Here $\varepsilon = M^{-1/3}$ ($M = |\sigma_T|A_1L^2\rho^{-1}\nu^{-2}$ is the Marangoni number, A_1 is a characteristic scale of the temperature gradient, and L is a characteristic horizontal scale). We denote by $U_1 = (|\sigma_T|^2A_1{}^2L\rho^{-2}\nu^{-1})^{1/3}$ the characteristic velocity in the boundary layer; we then introduce the dimensionless pressure p' according to the equation $p = \rho U^2p' - \rho gz$. We take $B_1 = \rho gh^2\sigma_0{}^{-1}$ as the Bond number. We calculate planar thermocapillary flow generated in the domain bounded by the free boundary Γ and by a solid wall due to heating of the free boundary at a given constant wall temperature. Asymptotic expansions of the solution of problem (1.1), (1.2), (Γ_1 is the empty set) in the limit $\varepsilon \to 0$ are constructed in the series form

$$\mathbf{v} \sim \mathbf{h}_0 + \varepsilon\mathbf{h}_1 + \ldots,\ \zeta \sim \varepsilon\zeta_1 + \ldots,\ p' \sim q_0 + \varepsilon q_1 + \ldots,$$
$$T \sim \tau_0 + \varepsilon\tau_1 + \ldots \tag{5.1}$$

The equations for the principal terms of the asymptotic expansions are derived by means of the second iteration process of the Vishik–Lyusternik method [7]. Equations (5.1), unlike the series (1.3) do not contain coefficients characterizing the external solution, because the entire flow domain coincides exactly with the boundary-layer domain. Let $\psi(s, x)$ be the stream function ($s = z/\varepsilon$); then $h_{x0} = \partial\psi/\partial s$, $h_{z0} = 0$, and $h_{z2} = -\partial\psi/\partial x$. The boundary-value problem for the principal terms of the expansions (5.1) are reduced to the form

$$\frac{\partial\psi}{\partial s}\frac{\partial^2\psi}{\partial x\,\partial s} - \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial s^2} = \frac{\partial^3\psi}{\partial s^3} - \frac{\partial q_0}{\partial x},\quad \frac{\partial q_0}{\partial s} = 0,\quad \frac{\partial\psi}{\partial s}\frac{\partial\tau_0}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\tau_0}{\partial s} = \mathrm{Pr}^{-1}\frac{\partial^2\tau_0}{\partial s^2},$$
$$\partial^2\psi_0/\partial s^2 = \partial\tau_\Gamma/\partial x,\ \psi = 0,\ \tau = \tau_\Gamma(x)\ (s = \zeta_1(x)),$$
$$\partial\psi/\partial x = \partial\psi/\partial s = 0,\ \tau_0 = \mathrm{const}\ (s = 0). \tag{5.2}$$

Here $\tau_\Gamma(x)$ is the temperature distribution along the free boundary. From the dynamical boundary condition on the free boundary we deduce the relation $\lambda q_0 = B\zeta_1$ ($\lambda = |\sigma_T|A_1L\sigma_0{}^{-1}$ is the dimensionless temperature gradient).

The system (5.2) admits a self-similar solution. Let the temperature distribution along the free boundary be given by the power law $\tau_\Gamma = -1.25\tau x^{4/5}$. It is readily shown that

$$\psi = -x^{-3/5}\tau^{1/3}b^2\Phi(\xi),\ \zeta_1 = b\tau^{-1/3}x^{2/3},\ \xi = 1 - sx^{-2/5}\tau^{1/3}b^{-1}.$$

The function $\Phi(\xi)$ and the parameter $b(\tau)$ are determined by solving the boundary-value problem

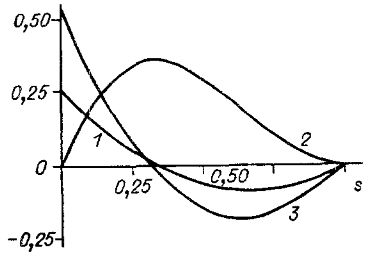

Fig. 4

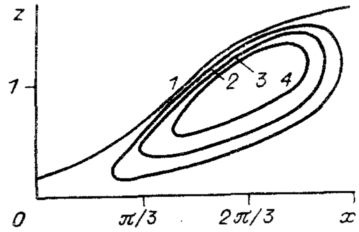

Fig. 5

$$5\Phi''' - b^3\Phi'^2 + 3b^3\Phi\Phi'' = 5\alpha b^2,\ \Phi(0) = \Phi(1) = \Phi'(1) = 0,\ \Phi''(0) = -1 \tag{5.3}$$

($\alpha = 2\lambda^{-1}\tau^{-5/3}/3$). Problem (5.3) is solved numerically. For $\alpha = 1$ we obtain the values $b = 1.218$ and $\Phi'(0) = 0.249$.

Figure 4 shows profiles of the longitudinal velocity $\Phi'(\xi)$ (curve 1) and the function $10\Phi(\xi)$ (curve 2). The velocity attains a maximum at the free boundary. A counterflow zone with a maximum velocity roughly one third smaller than the velocity at the free boundary occurs in the domain $0 \le z \le 0.66\zeta_1$.

Problem (5.2) is solved by Galerkin's method for the case of a harmonic temperature distribution $\tau_\Gamma = \tau_a \sin x$. The unknown flow domain D_Γ is mapped into a rectangular strip by the transformation $\eta = 1 - s\zeta^{-1}(x)$. The stream function and the elevation of the free boundary are represented by the series

$$\psi = \sum_{k=0}^{N} \psi_k(\eta) \sin kx, \quad \zeta_1 = \zeta_{10} + \sum_{k=1}^{N} \zeta_{1k} \cos kx.$$

The coefficients ψ_k and ζ_{1k} satisfy a system of ordinary differential equations, which are solved numerically by the regula falsi method. In the calculations it is assumed that $N = 3$, $\lambda = 0.143$, and $L = 5$. The average thickness of the layer is 0.1 cm, and $B = 0.055$. This corresponds to a temperature gradient $A_1 = 10$ deg/cm and a parameter $\varepsilon = 0.0126$.

Figure 5 shows the pattern of streamlines for $\tau_a = 0.2$ in one cell. The streamlines in the interval $[\pi, 2\pi]$ are symmetrical about the line $x = \pi$. Curve 1 represents the shape of the free boundary, and curves 2-4 are the streamlines $\psi = c$ at $c = 0.002, 0.004, 0.006$. Curve 3 in Fig. 4 represents the profile of the velocity $-10h_{x0}$ in the cross section $x = 0.5\pi$. The calculations show that the velocity reaches a maximum at the free boundary. Along this boundary the velocity increases from zero at $x = 0$ to the maximum value and then decreases to zero at $x = \pi$. As the amplitude of the temperature τ_a increases, the thickness of the layer decreases at the point $x = 0$ and increases at the point $x = \pi$. For $\tau_a = \tau_* = 0.28$ the free boundary comes into contact with the solid wall at the point $x = 0$. For $\tau_a > \tau_*$ the liquid layer divides into strips, which wet the solid wall in the zones $\pi - \alpha + 2\pi m \le x \le \pi + \alpha + 2\pi m$ ($m = 0, \pm 1, \ldots$); the wall is not wetted outside these zones. The values of α depend on τ_a and are determined numerically; for example, $\alpha = 0.144$ for $\tau_a = 0.35$.

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DROPLET SIZE DISTRIBUTION IN A PERCOLATION MODEL FOR EXPLOSIVE LIQUID DISPERSAL

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A very complicated problem on irregular motion is involved in theoretical analysis of liquid deformation during dispersal and explosive break-up, which in general cannot be solved accurately. The chaotic (explosive) dispersal of a liquid is here related to the break-up in an infinite cluster as studied in percolation theory. The drop size distribution is derived theoretically. If the dispersal is planar, the standard empirical relations are obtained (the Rozin—Rammler law and Weibull distribution), but in the three-dimensional case, there are deviations from them. Measurements have also been made on dispersal for a concentrated elastoviscous liquid based on a polymer on wire explosion in a cylindrical volume. The measurements on the whole agree satisfactorily with the theory.

1. The following liquids are examined here: Newtonian ones (in particular, ideal ones) with surface tension and polymer ones, which have internal entropy elasticity [1]. For a sufficiently concentrated polymer liquid (≳1%), the surface tension is usually unimportant, since virtually always $Ga_0/\alpha \gg 1$, in which G is the elastic modulus, α the surface tension, and a_0 the minimum characteristic dimension, which is discussed in detail below.

At t = 0, a bounded liquid volume with characteristic dimension R_0 acquires kinetic energy E_0 due for example to an explosion at the center. This concerns particularly the electrical explosion of a wire or a detonation within a bounded volume (see [2-4] and Sec. 3 below). In such cases, there are several factors that lead almost instantly to irregular chaotic deformation, which precedes the break-up and favors the latter. One of them is the shape imperfection or inhomogeneity in the exploding wire or detonator, which leads to initial irregularity in the velocity pattern. Another is that the explosion-cavity expansion is accompanied by Rayleigh—Taylor instability [5-7], which is the first stage in the irregular motion. As that form of instability develops, the motion becomes more complicated and chaotic, and in the nonlinear stage of perturbation growth, vortices arise at the tips of the fingers. To some extent, the break-up itself indicates that there are irregular motions, and accentuation of the chaotic motion is evident at the stage where there are separate droplets, which is evidently due to new modes occurring, particularly on expansion in the vacuum, which can occur in the motion of the continuous volume at least as small perturbations.

Sometimes, one expects that the kinetic energy in the irregular motions arising from a central explosion will be $E \sim E_0$; this is evident from estimates of the dissipative losses. Also, in general refining E does not affect the theory and merely has a quantitative effect on the results.

We assume that the central explosion almost immediately gives rise to internal deformations corresponding to many degrees of freedom, which absorb much of E_0. Such motion has been discussed [8] and is due either to turbulence or to initial inhomogeneity in the velocity pattern from the explosion.

In general, the dispersal and explosive break-up at present do not allow of a formal discussion of the internal irregular motion excitation. Also, there are some examples where

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